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AN OPTIMAL LINEAR FEEDBACK GUIDANCE SCHEME

Stuart E. Dreyfus and Jarrell R. Elliott

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PREFACE

The research reported in this Memorandum was begun under sponsorship of the National Aeronautics and Space Administration. A considerable portion of the work was conducted at the Langley Research Center, Langley Air Force Base, Hampton, Virginia, where Mr. Elliott is a NASA aerospace engineer. Mr. Dreyfus, of The RAND Corporation, participated in the project while on a leave of absence from RAND. The project was completed and the manuscript prepared following Mr. Dreyfus' return to RAND, his share of that work being sponsored by U.S. Air Force Project RAND.

The linear feedback guidance scheme developed simplifies the problem of correcting space craft inflight disturbances, which is particularly important in space vehicle rendezvous applications and initiating post-launch flight corrections.

NASA has agreed to publication of the work in this form.

SUMMARY

A theory developing a linear feedback guidance scheme to correct for inflight disturbances of a vehicle during the course of a space mission is presented. The theory is predicated on the use of a nominal optimal trajectory. The scheme consists of a linear combination of (1) perturbations of the vehicle state from its nominal state, and (2) time-varying gains to determine the control correction required to satisfy the constraints of the trajectory in an optimal fashion. Exact knowledge of the state of the vehicle is assumed.

An analysis of numerical results for an idealized rocket trajectory problem shows the linear feedback guidance scheme to be effective over a wide range of chosen perturbations.

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I. INTRODUCTION

Suppose that, for a particular vehicle and space mission, an optimal trajectory has been determined. Let us call this optimal trajectory, associated with the specified initial conditions and computed before launch, the nominal trajectory. Suppose further that due to abnormalities during flight the vehicle deviates from this trajectory. For the specified mission there is now a new optimal trajectory associated with the new state (i.e., position, velocity, mass, etc.) of the vehicle, and neither a return to the nominal trajectory nor continued use of the originally programmed control is desirable.

This Memorandum develops a simple linear feedback guidance scheme which determines the control correction necessary to yield a "corrected" optimal trajectory if the vehicle is slightly perturbed from its original optimal trajectory. Exact measurement of the state of the vehicle at discrete time intervals is assumed. The control correction rule is a linear function of the state variable deviations from nominal with time-varying coefficients that are computed and stored before flight.

The technique of derivation is akin to dynamic programming [1]. We shall define an auxiliary function of the state variables to be called the optimal return

I. INTRODUCTION

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The technique of derivation is akin to dynamic programming [1]. We shall define an auxiliary function of the state variables to be called the optimal return function, characterize it by means of the principle of optimality, and then deduce the guidance scheme from the resulting relations. We shall approximate the optimal return function in the neighborhood of a given optimal nominal trajectory rather than compute the optimal return function for all state space. This is accomplished by computing the first and second partial derivatives of the optimal return function associated with the nominal trajectory. Thus, at least to first order, we obtain the optimal return function for any state slightly perturbed from the nominal trajectory. For large deviations from nominal, the approximation is in error and the guidance scheme is inaccurate. The meaning of "small deviation" depends on the extent of nonlinearity of any particular problem and can best be determined by numerical experimentation.

The problem stated above has also been studied by Breakwell and Bryson [2] and by Kelley [3]. Their mathematical approaches differ considerably from ours, and the computation of their linear guidance rule involves quite different numerical operations. Presumably, however, the end results are mathematically equivalent. Questions of comparative numerical efficiency and accuracy of the three methods remain to be investigated.

II. THE PROBLEM

The optimal trajectory problem we shall consider has the following discrete-time statement. Find that sequence of controls,

$$\{u(t)\}, \ t = t_o, \ t_o + \Delta, \ t_o + 2\Delta, \ - - - \tag{2-1}$$

such that the dynamical system governed by the difference equations

$$x_i(t + \Delta) = x_i(t) + f_i\Big(x_1(t), \ . \ . \ . \ , \ x_{N-1}(t), \ t, u(t)\Big)\Delta, \ i = 1, \ . \ . \ . \ , \ N - 1 \tag{2-2}$$

$$x_i(t_o) = x_{i_o}, \ i = 1, \ . \ . \ ., \ N - 1 \tag{2-3}$$

evolves in such a way that the function

$$\varphi\Big(x_1(t), \ . \ . \ ., \ x_{N-1}(t), \ t\Big) \tag{2-4}$$

is minimized at some unspecified time t determined by the satisfaction of the m terminal conditions

$$\psi_j\Big(x_1(t), \ . \ . \ ., \ x_{N-1}(t), \ t\Big) = 0 \tag{2-5}$$

$$j = 1, \ . \ . \ ., \ m < N.$$

That is, we want to determine a control function $u(t)$ that minimizes an objective function $\varphi\left(x_1, \ldots, x_{N-1}, t\right)$ at a terminal time determined by the simultaneous satisfaction of m terminal conditions. For the sake of simplicity of notation we define and use

$$x_N = t$$

$$f_N = 1.$$

We state the problem discretely since it must be solved digitally and used mechanically in a discrete way. Analogous continuous results are easily obtained by letting Δ approach zero.

III. THE GUIDANCE EQUATIONS

We assume that associated with any set of feasible initial conditions x_{i_o}, $i = 1, \ldots, N$, there exists an optimal trajectory to the desired terminal state and an associated minimal value of φ at the end point. Let us define the optimal return function $S\left(x_{1_o}, \ldots, x_{N_o}\right)$ by

$S\left(x_{1_o}, \ldots, x_{N_o}\right) =$ the value of φ at the terminal time determined by $\psi_j = 0$, where the vehicle starts in state $x_{1_o}, \ldots, x_{N_o}$ and an optimal trajectory is followed.

The function $S\left(x_{1_o}, \ldots, x_{N_o}\right)$ satisfies the relation

$$S\left(x_{1_o}, \ldots, x_{N_o}\right) \tag{3-1}$$

$$= \underset{u_o}{\mathrm{Min}}\left[S\left(x_{1_o} + f_1\left(x_{1_o}, \ldots, x_{N_o}, u_o\right)\Delta, \ldots, x_{N_o} + f_{N_o}\left(x_{1_o}, \ldots, x_{N_o}, u_o\right)\Delta\right)\right]$$

where $u_o = u(t_o) = u\left(x_{N_o}\right)$. To simplify notation we write f_{i_o} for $f_i\left(x_{1_o}, \ldots, x_{N_o}, u_o\right)$.

Equation (3-1) is a statement of the fact that the minimal terminal value of φ (i.e., optimal return) associated with a given state at time t_o equals the optimal return associated with the state at time $(t_o + \Delta)$ yielded by the control action u at time t_o where $u(t_o)$ has been so chosen as to yield that state at time $(t_o + \Delta)$ with minimal return. This reasoning is called, by Bellman, the principle of optimality.

The above characterization of S implies two equations. The first of these is a necessary condition for optimality; i.e., that the right-hand side of (3-1) be a minimum with respect to all admissible controls u_o. If u is unbounded, we minimize by differentiation and obtain

$$\sum_{i=1}^{N} \frac{\partial S\left(x_{1_o} + f_{1_o}\Delta, \ldots, x_{N_o} + f_{N_o}\Delta\right)}{\partial\left(x_{i_o} + f_{i_o}\Delta\right)} \frac{\partial f_{i_o}}{\partial u_o} = 0 . \quad (3\text{-}2)$$

In equation (3-2) we have used the chain rule for the derivative of a function whose arguments are, in turn, functions. The second equation is a statement of the fact that, on the optimal trajectory from any initial state, the value S is invariant, regardless of the particular initial state. That is

$$S\left(x_{1_o}, \ldots, x_{N_o}\right) = S\left(x_{1_o} + f_{1_o}\Delta, \ldots, x_{N_o} + f_{N_o}\Delta\right) \quad (3\text{-}3)$$

where the f_{i_o} are evaluated using the optimal u_o. Note that the optimal u_o, u_o^*, defined by equation (3-2) depends on the point in state variable space since both $\frac{\partial S}{\partial x_{i_o}}$ and f_{i_o} depend on the state.

We now wish to consider the nominal trajectory, which is the optimal trajectory from a particular specified initial state. If we think of time as indexing each point in state space along this trajectory, S has the same contant value at each time.

$$\frac{\partial S}{\partial x_{i_o}}, \frac{\partial^2 S}{\partial x_{i_o}^2}, \text{etc.},$$

have numerical values which depend on the time, and the combination of the derivatives defined in (3-2) is zero.

For convenience and to simplify notation we use double subscript notation to denote summation. Also since we will be concerned with adjacent states in state variable space (i.e., proceeding discretely in time) we drop the subscript o and use the following notation to denote evaluation of the partials at the appropriate time:

$$\left.\frac{\partial S(x_1 + f_1\Delta, \ldots, x_N + f_N\Delta)}{\partial(x_i + f_i\Delta)}\right|_{x_N = t} = \left.\frac{\partial S}{\partial x_i}\right|_{t+\Delta} .$$

Rewriting equations (3-2) and (3-3) in this notation we have

$$\frac{\partial S}{\partial x_i}\bigg|_{t+\Delta} \frac{\partial f_i}{\partial u}\bigg|_t = 0 \tag{3-4}$$

$$S\,\bigg|_t = S\,\bigg|_{t+\Delta} . \tag{3-5}$$

Our object is to determine $\left.\frac{\partial u^*}{\partial x_j}\right|_t$, the rate of change in optimal control u^* associated with a perturbation in the nominal value of x_j at time t. Since equation (3-4) determines the optimal control, we take its partial derivative with respect to x_j keeping in mind that the optimal control decision u^* may change due to this perturbation in x_j. This yields

$$\frac{\partial^2 S}{\partial x_i \partial x_j}\bigg|_{t+\Delta} \frac{\partial f_i}{\partial u}\bigg|_t + \frac{\partial^2 S}{\partial x_i \partial x_k}\bigg|_{t+\Delta} \left[\frac{\partial f_k}{\partial x_j}\bigg|_t \right. \tag{3-6}$$

$$\left. + \frac{\partial f_k}{\partial u}\bigg|_t \frac{\partial u^*}{\partial x_j}\bigg|_t\right] \frac{\partial f_i}{\partial u}\bigg|_t \Delta + \frac{\partial S}{\partial x_i}\bigg|_{t+\Delta} \left[\frac{\partial^2 f_i}{\partial u \partial x_j}\bigg|_t + \frac{\partial^2 f_i}{\partial u^2}\bigg|_t \frac{\partial u^*}{\partial x_j}\bigg|_t\right] = 0$$

which gives $\frac{\partial u^*}{\partial x_j}$ in terms of the first and second partial derivatives of S evaluated on the nominal trajectory.

To determine a recurrence relation for $\frac{\partial S}{\partial x_j}$ on the nominal trajectory we take the partial derivative of (3-5) with respect to x_j. This yields

$$\left.\frac{\partial S}{\partial x_j}\right|_t = \left.\frac{\partial S}{\partial x_j}\right|_{t+\Delta} + \left.\frac{\partial S}{\partial x_i}\right|_{t+\Delta} \left.\frac{\partial f_i}{\partial x_j}\right|_t \Delta \tag{3-7}$$

where we have used the fact that

$$\left.\frac{\partial S}{\partial x_i}\right|_{t+\Delta} \left.\frac{\partial f_i}{\partial u}\right|_t$$

is zero (see equation (3-4)).

To obtain a recurrence relation for $\frac{\partial^2 S}{\partial x_j \partial x_l}$ on the nominal trajectory we differentiate (3-7) with respect to x_l, obtaining

$$\begin{aligned}\left.\frac{\partial^2 S}{\partial x_j \partial x_l}\right|_t &= \left.\frac{\partial^2 S}{\partial x_j \partial x_l}\right|_{t+\Delta} + \left.\frac{\partial^2 S}{\partial x_j \partial x_k}\right|_{t+\Delta} \left[\left.\frac{\partial f_k}{\partial x_l}\right|_t + \left.\frac{\partial f_k}{\partial u}\right|_t \left.\frac{\partial u^*}{\partial x_l}\right|_t\right] \Delta + \left.\frac{\partial^2 S}{\partial x_i \partial x_l}\right|_{t+\Delta} \left.\frac{\partial f_i}{\partial x_j}\right|_t \Delta \\ &+ \left.\frac{\partial^2 S}{\partial x_i \partial x_k}\right|_{t+\Delta} \left[\left.\frac{\partial f_k}{\partial x_l}\right|_t + \left.\frac{\partial f_k}{\partial u}\right|_t \left.\frac{\partial u^*}{\partial x_l}\right|_t\right] \left.\frac{\partial f_i}{\partial x_j}\right|_t \Delta^2\end{aligned} \tag{3-8}$$

$$+ \frac{\partial S}{\partial x_i}\Bigg|_{t+\Delta} \left[\frac{\partial^2 f_i}{\partial x_j \partial x_l}\Bigg|_t + \frac{\partial^2 f_i}{\partial u \partial x_j}\Bigg|_t \frac{\partial u^*}{\partial x_l}\Bigg|_t \right] \Delta \quad .$$

Given the first and second partial derivatives of S at time $t + \Delta$ along the nominal trajectory, equation (3-6) allows us to compute $\frac{\partial u^*}{\partial x_j}$ at time t. Equations (3-7) and (3-8) yield the first and second partial derivatives of S at time t thus allowing the continuation of the backward recursion.

Once we know $\frac{\partial u^*}{\partial x_j}$, the change in optimal control dictated by a change in state, as a function of time along a nominal trajectory, the linear rule

$$\delta u^*(t) = \sum_{j=1}^{N} \frac{\partial u^*}{\partial x_j}(t)\delta x_j(t) \tag{3-9}$$

yields the change in u^* required to correct for deviations from nominal, δx_j, in the state variables.

This is the linear feedback guidance scheme. The time-varying coefficient functions in (3-9), $\frac{\partial u^*}{\partial x_j}(t)$, are computed along the nominal trajectory before flight by recursion of equations (3-6), (3-7), and (3-8), and stored in the guidance computer. During flight, at time $t_o + k\Delta$ when the states are observed, the state deviations from nominal are multiplied by the proper stored coefficients and summed, as indicated by equation (3-9), to obtain the proper adjustment, $\delta u^*(t)$, in control.

In the next section we derive the terminal values of $\frac{\partial S}{\partial x_i}$ and $\frac{\partial^2 S}{\partial x_i \partial x_k}$ that allow the backward recursion of equations (3-6), (3-7), and (3-8).

IV. BOUNDARY VALUES

The proposed guidance scheme involves the computation of the second partial derivatives of S backward from the terminal time along a nominal (optimal) trajectory as well as the necessary partial derivatives of the control. The terminal values can be deduced by at least two methods. Method I yields some physical insight into the nature of the limiting process involved; however, it is cumbersome and impractical for complicated problems. Method II appears to be more practical and is recommended for general usage.

Method I--A set of terminal equations

$$\Psi_j(x_1, \ldots, x_N) = 0 \quad j = 1, \ldots, m \qquad (4-1)$$

is satisfied at the final time T by the nominal trajectory. Let a change from nominal in any state at a time t shortly before T occur. The control must then be adjusted so that the m constraining equations remain satisfied at the, perhaps changed, terminal time.

Let us evaluate Ψ_j at terminal time T by means of a Taylor series expansion about time t shortly before T. Thus

$$\Psi_j(T) = \Psi_j(t) + (T - t)\dot{\Psi}_j(t) + \frac{(T - t)^2}{2}\ddot{\Psi}_j(t) + \ldots \qquad (4-2)$$

The fact that $\Psi_j(T)$, where T is the terminal time which may change if we change state at time t, must equal 0 at the terminal time, yields the m equations

$$\frac{\partial \Psi_j(T)}{\partial x_i(t)} = 0 = \frac{\partial \Psi_j(t)}{\partial x_i(t)} + \frac{\partial T}{\partial x_i(t)} \dot{\Psi}_j(t) \qquad (4\text{-}3)$$

$$+ (T - t)\left[\frac{\partial \dot{\Psi}_j(t)}{\partial x_i(t)} + \frac{\partial \dot{\Psi}_j}{\partial u(t)} \frac{\partial u^*(t)}{\partial x_i(t)}\right] + \ldots \qquad j = 1, \ldots, m$$

corresponding to state x_i.

Since $\dot{\Psi}_j$ depends on the control u^*, $\ddot{\Psi}_j$ upon u^* and $\dot{u}^*$, etc., the above m linear equations can be solved for the m quantities

$$\frac{\partial T}{\partial x_i(t)}, \frac{\partial u^*(t)}{\partial x_i(t)}, \ldots, \frac{\partial u^*(t)^{(m-1)}}{\partial x_i(t)} \qquad (4\text{-}4)$$

in terms of the nominal state variables and control.

Now, writing the objective function φ, similarly, as

$$\varphi(T) = \varphi(t) + (T - t)\dot{\varphi}(t) + \ldots, \qquad (4\text{-}5)$$

we have

$$\frac{\partial\varphi(T)}{\partial x_i(t)} = \frac{\partial\varphi(t)}{\partial x_i(t)} + \frac{\partial T}{\partial x_i(t)}\,\dot{\varphi}(t) \qquad (4\text{-}6)$$

$$+ (T - t)\left[\frac{\partial\dot{\varphi}(t)}{\partial x_i(t)} + \frac{\partial\dot{\varphi}(t)}{\partial u(t)}\,\frac{\partial u^*(t)}{\partial x_i(t)}\right] + \ldots .$$

Since the left side of equation (4-6) is by definition

$$\left.\frac{\partial S}{\partial x_i}\right|_t$$

we now have the values of the first partial derivatives of S along a nominal trajectory at a time t just before the terminal time T. As $t \to T$

$$\frac{\partial u(t)^*}{\partial x_i(t)},\ \frac{\partial \dot{u}(t)^*}{\partial x_i(t)},$$

etc., in expression (4-4) approach infinity, so we begin the backward recursion a small time Δ before the terminal time. Then these quantities are large but finite.

To obtain the second partial derivatives of S we take the partial derivative of (4-6) after substituting for

$$\frac{\partial T}{\partial x_i(t)},\ \frac{\partial u^*(t)}{\partial x_i(t)},$$

etc., their known equivalents in terms of state and control variables. This gives expressions for the second partial derivatives of S near the endpoint and equations (3-6), (3-7), and (3-8) can now be solved by backward recursion.

As will be seen in subsequent sections where the above scheme is illustrated, the algebra involved in this method of evaluating the terminal boundary values is tedious.

Method II--This method is an adaptation of the work of Breakwell and Bryson [2]. It uses a portion of their scheme for optimal linear feedback guidance to determine the necessary partial derivatives of the control with respect to the state variables and those terminal second partial derivatives necessary in our approach.

The following results hold at the terminal point of a nominal (optimal) trajectory.

$$\Psi_j = \Psi_j(x_1,\ldots,x_N) = 0 \qquad j = 1,\ldots,m \tag{4-7}$$

$$\frac{\partial S}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} + \nu_j \frac{\partial \Psi_j}{\partial x_i} \qquad i = 1,\ldots,N \tag{4-8}$$

$$\frac{\partial S}{\partial x_i} f_i = 0 . \tag{4-9}$$

The ν_j above are m auxiliary numbers produced during the computation process of determining the nominal trajectory [4]. On a neighboring optimal trajectory (which is within some region where the assumption of linearity is valid) that satisfies the constraints of the problem, the following $N + m + 1$ equations hold. (Time is assumed to be the independent variable in the following.)

$$\frac{\partial \Psi_j}{\partial x_i} \delta x_i + \dot{\Psi}_j \, dt_f = 0 \qquad j = 1, \ldots, m \tag{4-10}$$

$$\delta \frac{\partial S}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial S}{\partial x_i}\right) dt_f = \frac{\partial^2 \varphi}{\partial x_i \partial x_k} \delta x_k + \nu_j \frac{\partial^2 \Psi_j}{\partial x_i \partial x_k} \delta x_k \tag{4-11}$$

$$+ \, d\nu_j \frac{\partial \Psi_j}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial \varphi}{\partial x_i} + \nu_j \frac{\partial \Psi_j}{\partial x_i}\right) dt_f \qquad i = 1, \ldots, N$$

$$\delta \frac{\partial S}{\partial x_i} f_i + \frac{\partial S}{\partial x_i} \frac{\partial f_i}{\partial x_k} \delta x_k = 0 \, . \tag{4-12}$$

The evaluation of the terms in the above three equations is made at the final time on the nominal trajectory. It should be noted that use of the fact that

$$\frac{\partial S}{\partial x_i} \frac{\partial f_i}{\partial u} = 0$$

has been made in equation (4-12). The $d\nu_j$ and $\delta\left(\frac{\partial S}{\partial x_i}\right)$ are the changes in ν_j and $\frac{\partial S}{\partial x_i}$, at the terminal time t_f of the nominal trajectory, required to satisfy the constraints of the problem in the additional time dt_f due to the perturbations δx_i at nominal terminal time t_f. Thus we have $N + m + 1$ equations in $2N + m + 1$ unknowns. Hence, there are N independent variables and the remaining variables are considered dependent. The equations relating the independent variables to the dependent variables are determined at the terminal time, t_f, of the nominal trajectory and used as boundary conditions for the backward integration of the linearized equations applicable to an optimal trajectory; that is, the differential equations of motion and the differential equations for the multipliers (here $\frac{\partial S}{\partial x_i}$). Also in order to solve for δu, the necessary condition for optimality is linearized. These equations are

$$\frac{d}{dt}(\delta x_i) = \frac{\partial f_i}{\partial x_j}\,\delta x_j + \frac{\partial f_i}{\partial u}\,\delta u \tag{4-13}$$

$$\frac{d}{dt}\left(\delta\,\frac{\partial S}{\partial x_i}\right) = -\left[\frac{\partial S}{\partial x_j}\,\frac{\partial^2 f_j}{\partial x_i \partial x_k}\,\delta x_k + \frac{\partial f_j}{\partial x_i}\left(\delta\,\frac{\partial S}{\partial x_j}\right) + \frac{\partial S}{\partial x_j}\,\frac{\partial^2 f_j}{\partial x_i \partial u}\,\delta u\right] \tag{4-14}$$

$$\frac{\partial S}{\partial x_j}\,\frac{\partial^2 f_j}{\partial u^2}\,\delta u = -\left[\frac{\partial S}{\partial x_j}\,\frac{\partial^2 f_j}{\partial u \partial x_k}\,\delta x_k + \frac{\partial f_j}{\partial u}\left(\delta\,\frac{\partial S}{\partial x_j}\right)\right]. \tag{4-15}$$

The linearized equations are solved backward in time N times, where each time all but one of the variables at the terminal time are assumed zero. The value of one is assigned to the remaining term in turn until all N terms have been used in this manner. This will be clearer after examination of the example problem. Next we solve for $(\delta \frac{\partial S}{\partial x_i})$ at time t in terms of the δx_i at time t. Then holding all δx_i except δx_j zero, we solve for

$$\frac{\delta \frac{\partial S}{\partial x_i}}{\delta x_j} \quad \text{at time} \quad t.$$

This is the desired $\frac{\partial^2 S}{\partial x_i \partial x_j}$. In similar fashion we solve equation (4-15) for $\delta u\Big|_t$ in terms of $\delta x_i\Big|_t$ and interpret $\frac{\delta u}{\delta x_i}$ as $\frac{\partial u}{\partial x_i}$. We then use these values as the boundary values to initiate recurrence equations (3-6), (3-7), and (3-8).

V. A SIMPLE EXAMPLE

To illustrate the above concepts, let us study a simple problem with a known solution. We wish to program the direction $\gamma(t)$ so that a particle moving in a plane at a constant velocity V reaches a fixed point (x_1, y_1) in minimum time. Here, in our previous notation,

$$x(t + \Delta) = x(t) + (V \cos \gamma(t))\Delta \tag{5-1}$$

$$y(t + \Delta) = y(t) + (V \sin \gamma(t))\Delta$$

$$\varphi(x,y,t) = t$$

$$\Psi_1(x,y,t) = x - x_1 = 0$$

$$\Psi_2(x,y,t) = y - y_1 = 0$$

where φ is the objective function to be minimized and Ψ_1 and Ψ_2 are terminal conditions.

Starting at the point (x_o, y_o) we determine, in this case by inspection, that the minimum time trajectory is a straight line between (x_o, y_o) and (x_1, y_1).

The optimal time of arrival at (x_1, y_1) starting at any initial point (x_o, y_o) at time t_o is then

$$S(x_o,y_o,t_o) = t_o + \frac{\sqrt{(x_1 - x_o)^2 + (y_1 - y_o)^2}}{V} . \tag{5-2}$$

To deduce the appropriate boundary conditions by Method I, we let

$$\Psi_1(T) = x(T) - x_1 = x(t) - x_1 + (T - t)V\cos\gamma \tag{5-3}$$

$$+ \frac{(T - t)^2}{2}(- V\sin\gamma\dot{\gamma}) + \ldots$$

$$\Psi_2(T) = y(T) - y_1 = y(t) - y_1 + (T - t)V\sin\gamma$$

$$+ \frac{(T - t)^2}{2}(V\cos\gamma\dot{\gamma}) + \ldots .$$

Then

$$\frac{\partial\Psi_1(T)}{\partial x(t)} = 0 = 1 + \frac{\partial T}{\partial x} V\cos\gamma + (T - t)(- V\sin\gamma)\frac{\partial\gamma}{\partial x} \tag{5-4}$$

$$\frac{\partial\Psi_2(T)}{\partial x(t)} = 0 = \frac{\partial T}{\partial x} V\sin\gamma + (T - t)\, V\cos\gamma\,\frac{\partial\gamma}{\partial x}\,, \tag{5-5}$$

neglecting higher order terms.

Hence,

$$\frac{\partial T}{\partial x} = -\frac{\cos\gamma}{V};\ \frac{\partial\gamma}{\partial x} = \frac{\sin\gamma}{V(T - t)}\,. \tag{5-6}$$

Then, since

$$\varphi(T) = T,$$

$$\frac{\partial S}{\partial x} = \frac{\partial T}{\partial x} = -\frac{\cos\gamma}{V} . \tag{5-7}$$

Similarly, replacing x by y in the above derivations,

$$\frac{\partial T}{\partial y} = -\frac{\sin\gamma}{V} \tag{5-8}$$

$$\frac{\partial \gamma}{\partial y} = -\frac{\cos\gamma}{V(T-t)}$$

$$\frac{\partial S}{\partial y} = -\frac{\sin\gamma}{V} .$$

These results are easily verified directly by differentiating (5-2). For example:

$$\frac{\partial S}{\partial x_o} = -\frac{(x_1 - x_o)}{V\sqrt{(x_1 - x_o)^2 + (y_1 - y_o)^2}} \tag{5-9}$$

$$= -\frac{\dfrac{x_1 - x_o}{T - t_o}}{V\sqrt{\left(\dfrac{x_1 - x_o}{T - t_o}\right)^2 + \left(\dfrac{y_1 - y_o}{T - t_o}\right)^2}} = -\frac{\dot{x}}{V\sqrt{\dot{x}^2 + \dot{y}^2}} = -\frac{\cos\gamma}{V}$$

in the limit as $t_o \to T$.

Continuing, we derive a typical terminal value of a second partial derivative of S. Partial differentiation of (5-7) with respect to y yields

$$\frac{\partial^2 S}{\partial x \partial y} = \frac{\sin \gamma \frac{\partial \gamma}{\partial y}}{V} = - \frac{\sin \gamma \cos \gamma}{V^2 (T - t)} .$$

Again, since the exact form of S is known in this example, this result can be verified:

$$\frac{\partial^2 S}{\partial x_o \partial y_o} = - \frac{(x_1 - x_o)(y_1 - y_o)}{V((x_1 - x_o)^2 + (y_1 - y_o)^2)^{3/2}} \tag{5-10}$$

$$= - \frac{\dot{x}\dot{y}}{V(T - t)V^3} = - \frac{\sin \gamma \cos \gamma}{V^2 (T - t)} .$$

For the purpose of illustrating the use of Method II, we observe as pointed out in Chap. V of Bellman and Dreyfus [1], that $\frac{\partial S}{\partial x_i}$ is equivalent to λ_{x_i}, the familiar multiplier functions of the classical calculus of variations. We also replace the difference equations (5-1) by differential equations and include time as a state of the system. Our system becomes

$$\begin{aligned} \dot{x} &= V \cos \gamma \qquad \dot{\lambda}_x = 0 \\ \dot{y} &= V \sin \gamma \qquad \dot{\lambda}_y = 0 \\ \dot{t} &= 1 \qquad \dot{\lambda}_t = 0 \end{aligned} \tag{5-11}$$

The optimal policy to be followed, γ^*, is given by

$$-\lambda_x V \sin\gamma + \lambda_y V \cos\gamma = 0 \Rightarrow \tan\gamma^* = \frac{\lambda_y}{\lambda_x}\,. \qquad (5\text{-}12)$$

From the terminal conditions of the optimal nominal trajectory we have (where superscript f denotes final value)

$$\lambda_x{}^f = \nu_1;\ \lambda_y{}^f = \nu_2;\ \lambda_t{}^f = 1\,. \qquad (5\text{-}13)$$

Since the time rate of change of the multipliers is zero we have

$$\lambda_x = \nu_1,\ \lambda_y = \nu_2 \text{ and } \lambda_t = 1\,. \qquad (5\text{-}14)$$

From the transversality condition at the terminal point and the optimal policy condition

$$\lambda_y{}^f = -\frac{\sin\gamma}{V} = \nu_2 \qquad (5\text{-}15)$$

$$\lambda_x{}^f = -\frac{\cos\gamma}{V} = \nu_1\,.$$

Then applying Method II to determine conditions on a neighboring optimal trajectory, equations (4-10), (4-11), and (4-12) become

$$\delta x + V\cos\gamma\, dt = 0 \qquad (5\text{-}16)$$

$$\delta y + V\sin\gamma\, dt = 0$$

$$\delta\lambda_x - d\nu_1 = 0$$

$$\delta\lambda_y - d\nu_2 = 0$$

$$\delta\lambda_t = 0$$

$$\delta\lambda_x V\cos\gamma + \delta\lambda_y V\sin\gamma + \delta\lambda_t = 0\,.$$

Since $N = 3$ and $m = 2$, we have $6 = N + m + 1$ equations in nine unknowns $(2N + m + 1)$. (The unknown δt does not appear since its coefficients are always zero.) We wish to choose three of the unknowns as independent such that the coefficients of the remaining six when written in matrix notation form a non-singular matrix. That is, we wish to form the matrix equation

$$AX = -F$$

and solve for X

$$X = -A^{-1}F \ .$$

In the above, we choose δt, $\delta\lambda_y$, and dt as independent. Choosing a convenient order we have,

$$X = \begin{bmatrix} \delta x \\ \delta y \\ d\nu_1 \\ d\nu_2 \\ \delta\lambda_t \\ \delta\lambda_x \end{bmatrix}_{t_f}; \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & V\cos\gamma \end{bmatrix}_{t_f} ;(5\text{-}17)$$

$$F = \begin{bmatrix} V\cos\gamma\ dt \\ V\sin\gamma\ dt \\ 0 \\ \delta\lambda_y \\ 0 \\ V\sin\gamma\ \delta\lambda_y \end{bmatrix}_{t_f}$$

and

$$A^{-1} = \frac{1}{V\cos\gamma}\begin{bmatrix} V\cos\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & V\cos\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & -V\cos\gamma & 0 & -1 & 1 \\ 0 & 0 & 0 & -V\cos\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & V\cos\gamma & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{t_f} . \quad (5\text{-}18)$$

Then writing the state variables and the multipliers in terms of the independent variables we have

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta t \end{bmatrix}_{t_f} = \begin{bmatrix} 0 & -V\cos\gamma & 0 \\ 0 & -V\sin\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_y \\ dt \\ \delta t \end{bmatrix}_{t_f} \quad (5\text{-}19)$$

$$\begin{bmatrix} \delta\lambda_x \\ \delta\lambda_y \\ \delta\lambda_t \end{bmatrix}_{t_f} = \begin{bmatrix} -\tan\gamma & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_y \\ dt \\ \delta t \end{bmatrix}_{t_f} .$$

Now the linearized differential equations (equations (4-13) and (4-14)) become

$$\frac{d}{dt}(\delta x) = V^2 \sin\gamma \left[\delta\lambda_y \cos\gamma - \delta\lambda_x \sin\gamma\right] \quad (5\text{-}20)$$

$$\frac{d}{dt}(\delta y) = -V^2 \cos\gamma \left[\delta\lambda_y \cos\gamma - \delta\lambda_x \sin\gamma\right]$$

$$\frac{d}{dt}(\delta t) = 0$$

$$\frac{d}{dt}(\delta\lambda_x) = \frac{d}{dt}(\delta\lambda_y) = \frac{d}{dt}(\delta\lambda_t) = 0$$

where

$$\delta\gamma = \frac{(\delta\lambda_y \cos\gamma - \delta\lambda_x \sin\gamma)}{\lambda_x \cos\gamma + \lambda_y \sin\gamma} \quad (5\text{-}21)$$

has been simplified and substituted into equations (5-20). Setting each of the independent variables, in turn, equal to 1 while the other two are set equal to 0 and integrating the linearized differential equations backward in time with initial conditions determined from the matrix equations (5-19), we have

$$\begin{bmatrix} \delta\lambda_x \\ \delta\lambda_y \\ \delta\lambda_t \end{bmatrix}_t = \begin{bmatrix} -\tan\gamma & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_y \\ dt \\ \delta t \end{bmatrix}_{t_f} \quad (5\text{-}22)$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta t \end{bmatrix}_t = \begin{bmatrix} -V^2\tan\gamma\,(t_f - t) & -V\cos\gamma & 0 \\ V^2(t_f - t) & -V\sin\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_y \\ dt \\ \delta t \end{bmatrix}_{t_f} . \quad (5\text{-}23)$$

Solving equation (5-22) for $\delta\lambda_y$, dt, and δt at time t_f in terms of the state variables at time t and inserting in equation (5-23), we have

$$\begin{bmatrix} \delta\lambda_x \\ \delta\lambda_y \\ \delta\lambda_t \end{bmatrix}_t = \frac{\cos\gamma}{V^3(t_f-t)} \begin{bmatrix} V\tan\gamma\sin\gamma & -V\sin\gamma & 0 \\ -V\sin\gamma & V\cos\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}_{t_f} \begin{bmatrix} \delta x \\ \delta y \\ \delta t \end{bmatrix}_t . \quad (5\text{-}24)$$

Now we let $\delta y = \delta t = 0$ and interpret $\frac{\delta\lambda_x}{\delta x} = \frac{\partial^2 S}{\partial x^2}$, etc. Thus we have

$$\frac{\partial^2 S}{\partial x^2} = \frac{\sin^2\gamma}{V^2(t_f - t)} \quad (5\text{-}25)$$

$$\frac{\partial^2 S}{\partial y^2} = \frac{\cos^2\gamma}{V^2(t_f - t)}$$

$$\frac{\partial^2 S}{\partial x\partial y} = -\frac{\sin\gamma\cos\gamma}{V^2(t_f - t)} .$$

Solving for $\delta\gamma$ (equation (5-21)) in terms of δx and δy at t in a similar manner, we have

$$\frac{\partial\gamma}{\partial x} = \frac{\sin\gamma}{V(t_f - t)} \quad (5\text{-}26)$$

$$\frac{\partial\gamma}{\partial y} = -\frac{\cos\gamma}{V(t_f - t)} .$$

VI. AN IDEALIZED PRACTICAL PROBLEM

To further illustrate and clarify the concepts developed here, an idealized practical problem has been solved. The problem is to determine the thrust attitude program of a rocket vehicle which will yield the maximum attainable horizontal velocity at specified terminal conditions of time, altitude, and vertical velocity. The following idealizing assumptions are made:

1) Vacuous atmosphere
2) Flat earth
3) Constant gravitational acceleration
4) Constant thrust acceleration.

The differential equations of the system are

$$\begin{aligned}
\dot{h} &= v & \dot{\lambda}_h &= 0 \\
\dot{v} &= a \sin\theta - g & \dot{\lambda}_v &= -\lambda_h \Rightarrow \lambda_v = -\lambda_h t + \lambda_{v_o} \\
\dot{u} &= a \cos\theta & \dot{\lambda}_u &= 0 \\
\dot{t} &= 1 & \dot{\lambda}_t &= 0
\end{aligned} \tag{6-1}$$

where

h = altitude

v = vertical velocity

u = horizontal velocity

a = thrust acceleration (constant)

g = gravitational acceleration (constant)

θ = angle of thrust acceleration above the horizontal

t = time.

It is well known that one can determine the form of the optimal policy for this problem from the necessary condition for optimality; that is

$$\tan \theta^* = \frac{\lambda_v}{\lambda_u} = -\lambda_h t + \lambda_{v_o} . \qquad (6\text{-}2)$$

The quantity to be maximized, φ, and the constraints, ψ_i, of the problem are

$$\begin{aligned} \varphi &= u \qquad (6\text{-}3) \\ \psi_1 &= h - h_f = 0 \\ \psi_2 &= v - v_f = 0 \\ \psi_3 &= t - t_f = 0 . \end{aligned}$$

For given initial conditions, an optimal trajectory can be determined which will satisfy the desired terminal conditions. Our problem then is to determine the optimal control action necessary to correct for small disturbances in this nominal trajectory. That is, we wish to determine the coefficients of δx in equation (3-9). For this, we need to determine the recurrence relations, equations (3-6), (3-7), and (3-8), and the boundary conditions required to initiate the recursion.

VII. RECURRENCE RELATIONS AND BOUNDARY VALUES

To derive recurrence relations for this problem, we define S by

$$S = \text{optimal } \varphi \text{ final} = \text{optimal } u(t_f) \ . \qquad (7\text{-}1)$$

We note that it is unnecessary to obtain the coefficient $\frac{\partial\theta}{\partial t}$ in our guidance rule since at time t_k a perturbation of time of δt from nominal can be regarded as a deviation in position and velocity from the nominal states at time $t_k + \delta t$. That is, any one state can be used to index points on the nominal and errors can then occur only in the other states. For rendezvous problems, time is not always the best indexing variable. Rewriting equations (6-1) as difference equations, we determine the recurrence equations for $\frac{\partial\theta}{\partial h}$, $\frac{\partial\theta}{\partial v}$, and $\frac{\partial\theta}{\partial u}$ to be

$$\left.\frac{\partial^2 S}{\partial h \partial v}\right|_{t+\Delta} \left. a \cos\theta \right|_t + k \left.\frac{\partial\theta}{\partial h}\right|_t = 0 \qquad (7\text{-}2)$$

$$\left.\frac{\partial^2 S}{\partial v^2}\right|_{t+\Delta} \left. a \cos\theta \right|_t + \Delta \left.\frac{\partial^2 S}{\partial h \partial v}\right|_{t+\Delta} \left. a \cos\theta \right|_t + k \left.\frac{\partial\theta}{\partial v}\right|_t = 0$$

$$k \left.\frac{\partial\theta}{\partial u}\right|_t = 0$$

where

$$k = \Delta \frac{\partial^2 S}{\partial v^2}\bigg|_{t+\Delta} a \cos^2\theta \bigg|_t - \frac{\partial S}{\partial u}\bigg|_{t+\Delta} a \cos\theta \bigg|_t \tag{7-3}$$

$$- \frac{\partial S}{\partial v}\bigg|_{t+\Delta} a \sin\theta \bigg|_t \quad .$$

Since $k \frac{\partial \theta}{\partial u} = 0$ we must have

$$\frac{\partial \theta}{\partial u} = 0 \; . \tag{7-4}$$

Therefore we need only concern ourselves with $\frac{\partial \theta}{\partial h}$ and $\frac{\partial \theta}{\partial v}$, deviations in horizontal velocity being irrelevant to the control action, as can easily be seen physically. The necessary recurrence relations are

$$\frac{\partial S}{\partial h}\bigg|_t = \frac{\partial S}{\partial h}\bigg|_{t+\Delta} \tag{7-5}$$

$$\frac{\partial S}{\partial u}\bigg|_t = 1$$

$$\frac{\partial S}{\partial v}\bigg|_t = \frac{\partial S}{\partial v}\bigg|_{t+\Delta} + \Delta \frac{\partial S}{\partial h}\bigg|_{t+\Delta}$$

$$\frac{\partial^2 S}{\partial h^2}\bigg|_t = \frac{\partial^2 S}{\partial h^2}\bigg|_{t+\Delta} + \frac{\partial^2 S}{\partial h \partial v}\bigg|_{t+\Delta} \left(a \cos\theta \, \Delta \frac{\partial \theta}{\partial h} \right) \bigg|_t$$

$$\left.\frac{\partial^2 S}{\partial h \partial v}\right|_t = \left.\frac{\partial^2 S}{\partial h \partial v}\right|_{t+\Delta} \left.\left(1 + a \cos\theta\, \Delta \frac{\partial \theta}{\partial v}\right)\right|_t + \Delta \left.\frac{\partial^2 S}{\partial h^2}\right|_{t+\Delta}$$

$$\left.\frac{\partial^2 S}{\partial v^2}\right|_t = \left.\left(\frac{\partial^2 S}{\partial v^2} + \Delta \frac{\partial^2 S}{\partial h \partial v}\right)\right|_{t+\Delta} \left.\left(1 + a \cos\theta\, \Delta \frac{\partial \theta}{\partial v}\right)\right|_t$$

$$+ \Delta \left.\left(\frac{\partial^2 S}{\partial h \partial v} + \Delta \frac{\partial^2 S}{\partial h^2}\right)\right|_{t+\Delta} .$$

The boundary conditions on $\frac{\partial \theta}{\partial h}$, $\frac{\partial \theta}{\partial v}$, and the necessary second partials of S have been determined by Method I, and verified by Method II. However, the use of Method II only is illustrated in the following. Here we again use the notation that

$$\frac{\partial S}{\partial x_i} = \lambda_{x_i} .$$

From the terminal conditions of the problem we have

$$\lambda_h{}^f = \nu_1,\ \lambda_v{}^f = \nu_2,\ \lambda_u{}^f = 1,\ \lambda_t{}^f = \nu_3 \qquad (7\text{-}6)$$

which, by virtue of the differential equations of the multipliers, implies

$$\lambda_h(t) = \nu_1,\ \lambda_v(t) = \tan\theta(t),\ \lambda_u(t) = 1, \qquad (7\text{-}7)$$
$$\lambda_t(t) = \nu_3 .$$

Application of equations (4-10), (4-11), and (4-12) results in

$$\delta h + v\,dt = 0 \tag{7-8}$$
$$\delta v + (a \sin\theta - g)dt = 0$$
$$\delta\lambda_h - d\nu_1 = 0$$
$$\delta\lambda_v - \lambda_h\,dt - d\nu_2 = 0$$
$$\delta\lambda_u = 0$$
$$\delta\lambda_t - d\nu_3 = 0$$
$$\delta\lambda_h v + \delta\lambda_v(a \sin\theta - g) + \delta\lambda_u\, a\cos\theta + \delta\lambda_t + \lambda_h\delta v = 0.$$

Choosing $\delta\lambda_h$, $\delta\lambda_v$, δu, and δt as independent and operating as before, we have

$$\begin{bmatrix} \delta h \\ \delta v \\ dt \\ d\nu_1 \\ d\nu_2 \\ \delta\lambda_u \\ d\nu_3 \\ \delta\lambda_t \end{bmatrix}_{t_f} = \begin{bmatrix} -1 & 0 & v & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_h & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda_h & -B\lambda_h & 0 & 0 & a\cos\theta & 1 & -1 \\ 0 & \lambda_h & -B\lambda_h & 0 & 0 & a\cos\theta & 0 & -1 \end{bmatrix}_{t_f} \begin{bmatrix} 0 \\ 0 \\ \delta t \\ \delta\lambda_h \\ \delta\lambda_v \\ 0 \\ 0 \\ v\delta\lambda_h + B\delta\lambda_v \end{bmatrix}_{t_f} \tag{7-9}$$

where $B = (a \sin\theta - g)$.

Writing the multipliers and the state variables at time t_f in terms of the independent variables, we have

$$\begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta\lambda_y \\ \delta\lambda_t \end{bmatrix}_{t_f} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v & -B & 0 & -B\lambda_h \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta u \\ \delta t \end{bmatrix}_{t_f} \qquad (7\text{-}10)$$

$$\begin{bmatrix} \delta h \\ \delta v \\ \delta u \\ \delta t \end{bmatrix}_{t_f} = \begin{bmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta u \\ \delta t \end{bmatrix}_{t_f} . \qquad (7\text{-}11)$$

The linearized differential equations are

$$\frac{d}{dt}(\delta h) = \delta v \qquad (7\text{-}12)$$

$$\frac{d}{dt}(\delta v) = a \cos^2 \theta(\delta\lambda_v \cos \theta - \delta\lambda_u \sin \theta)$$

$$\frac{d}{dt}(\delta u) = -a \sin \theta \cos \theta(\delta\lambda_v \cos \theta - \delta\lambda_u \sin \theta)$$

$$\frac{d}{dt}(\delta t) = 0$$

$$\frac{d}{dt}(\delta\lambda_h) = \frac{d}{dt}(\delta\lambda_u) = \frac{d}{dt}(\delta\lambda_t) = 0$$

$$\frac{d}{dt}(\delta\lambda_v) = -\delta\lambda_h$$

where

$$\delta\theta = \cos \theta \ (\delta\lambda_v \cos \theta - \delta\lambda_u \sin \theta) . \qquad (7\text{-}13)$$

Letting $\delta\lambda_h$, $\delta\lambda_v$, δu, and δt equal 1, in turn, while equating the others to 0 and integrating equations (7-12) backward in time with initial conditions from equations (7-10) and (7-11) we have

$$\begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta\lambda_u \\ \delta\lambda_t \end{bmatrix}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \bar{t} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v & -B & 0 & -B\lambda_h \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta u \\ \delta t \end{bmatrix}_{t_f} \tag{7-14}$$

$$\begin{bmatrix} \delta h \\ \delta v \\ \delta u \\ \delta t \end{bmatrix}_t = \begin{bmatrix} \dfrac{a \cos^3\theta\ \bar{t}^3}{6} & \dfrac{a \cos^3\theta\ \bar{t}^2}{2} & 0 & v-B\bar{t} \\ -\dfrac{a \cos^3\theta\ \bar{t}^2}{2} & -a \cos^3\bar{t} & 0 & B \\ \dfrac{a \sin\theta \cos^2\theta\ \bar{t}^2}{2} & a \sin\theta \cos^2\theta\ \bar{t} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{t_f} \begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta u \\ \delta t \end{bmatrix}_{t_f} \tag{7-15}$$

where $\bar{t} = t_f - t$.

Solving equations (7-14) and (7-15) for the delta multipliers at time t in terms of the delta state variables at time t, we have

$$\begin{bmatrix} \delta\lambda_h \\ \delta\lambda_v \\ \delta\lambda_u \\ \delta\lambda_t \end{bmatrix}_t = \frac{12}{a\cos^3\theta\,\bar{t}^3} \begin{bmatrix} -1 & -\frac{\bar{t}}{2} & 0 & v - \frac{B\bar{t}}{2} \\ -\frac{\bar{t}}{2} & -\frac{\bar{t}^2}{3} & 0 & \frac{v\bar{t}}{2} - \frac{B\bar{t}^2}{6} \\ 0 & 0 & 0 & 0 \\ v - \frac{B\bar{t}}{2} & \frac{v\bar{t}}{2} - \frac{B\bar{t}^2}{6} & 0 & v(B\bar{t}-v) - \frac{B\bar{t}^2}{3}\left(B + \frac{a\lambda_h\cos^3\theta\,\bar{t}}{4}\right) \end{bmatrix}_{t_f} \begin{bmatrix} \delta h \\ \delta v \\ \delta u \\ \delta t \end{bmatrix}_t . \quad (7\text{-}16)$$

Then letting $\frac{\delta\lambda_h}{\delta h} = \frac{\partial^2 S}{\partial h^2}$, $\frac{\delta\lambda_h}{\delta v} = \frac{\partial^2 S}{\partial h \partial v}$, etc., where δh, δv, δu, and δt are set equal to 1, in turn, we have

$$\left.\frac{\partial^2 S}{\partial h^2}\right|_t = -\frac{12}{a\cos^3\theta\,(t_f - t)^3} \quad (7\text{-}17)$$

$$\left.\frac{\partial^2 S}{\partial h \partial v}\right|_t = -\frac{6}{a\cos^3\theta\,(t_f - t)^2}$$

$$\left.\frac{\partial^2 S}{\partial v^2}\right|_t = -\frac{4}{a\cos^3\theta\,(t_f - t)} .$$

Evaluating $\delta\theta$ in equation (7-13) in a similar manner and letting $\frac{\delta\theta}{\delta h} = \frac{\partial\theta}{\partial h}$ and $\frac{\delta\theta}{\delta v} = \frac{\partial\theta}{\partial v}$, we find

$$\left.\frac{\partial\theta}{\partial h}\right|_t = -\frac{6}{a\cos\theta\,(t_f - t)^2} \quad (7\text{-}18)$$

$$\left.\frac{\partial\theta}{\partial v}\right|_t = -\frac{4}{a\cos\theta\,(t_f - t)} .$$

The remaining boundary values necessary to initiate the recursion are

$$\left.\frac{\partial S}{\partial h}\right|_t = \lambda_h \tag{7-19}$$

$$\left.\frac{\partial S}{\partial v}\right|_t = \tan \theta^*(t) \quad \text{(predetermined as a part of the nominal optimal trajectory)}$$

$$\left.\frac{\partial S}{\partial u}\right|_t = 1.$$

This analysis of the boundary values has been more general than necessary. However, it should serve to show the utility of the method.

VIII. RESULTS

The performance of the linear feedback guidance scheme was evaluated by comparing solutions using the scheme to correct for initial condition perturbations about a nominal (optimal) trajectory with true optimal trajectory solutions for the same perturbed initial conditions. The optimal trajectories were computed using the method of steepest ascent [4].

The nominal optimal trajectory solved the problem with the following constants and terminal conditions

$$a = 2g \text{ ft/sec}^2$$

$$g = 32 \text{ ft/sec}^2$$

$$h_f = 100{,}000 \text{ ft}$$

$$v_f = 0 \text{ ft/sec}$$

$$t_f = 100 \text{ sec.}$$

The initial conditions of the nominal trajectory at time $t = 0$ were

$$h_o = 0 \text{ ft}$$

$$v_o = 0 \text{ ft/sec}$$

$$u_o = 0 \text{ ft/sec.}$$

The accuracy of the computer program was verified by comparing nominal trajectory results from the computer program with the desired terminal conditions and the analytical solution for the optimal horizontal velocity, u. The accuracy was considered excellent.

Perturbations in the nominal trajectory were assumed by varying the initial conditions of the problem. Table I shows the various perturbations assumed and both the

Table I

Optimal Solution		Guidance Scheme Solution		
Case	u_f, ft/sec	h_f, ft	v_f, ft/sec	u_f, ft/sec
Nominal	3507.81			
$\delta h(o) = 1000$	3569.94	100,000	0.75	3570.60
$\delta h(o) = -1000$	3443.43	100,000	-0.44	3442.53
$\delta h(o) = 5000$	3796.14	100,001	6.70	3798.61
$\delta v(o) = 50$	3733.78	100,000	3.29	3736.11
$\delta v(o) = -50$	3246.40	100,000	-0.46	3243.94
$\delta t = 3$ sec	2960.43	100,001	0.72	2957.31

computed optimal solution and the guidance scheme solution based on the above nominal trajectory. (In all cases the computed optimal solution satisfied the desired terminal constraints within 0.1 ft in altitude and 0.1 ft/sec in vertical velocity.) The feedback gains used are shown plotted against time in Fig. 1. The last case shown, $\delta t = 3$ sec, corresponds to the case where 97 seconds of flight time remain; i.e., launch is three seconds late.

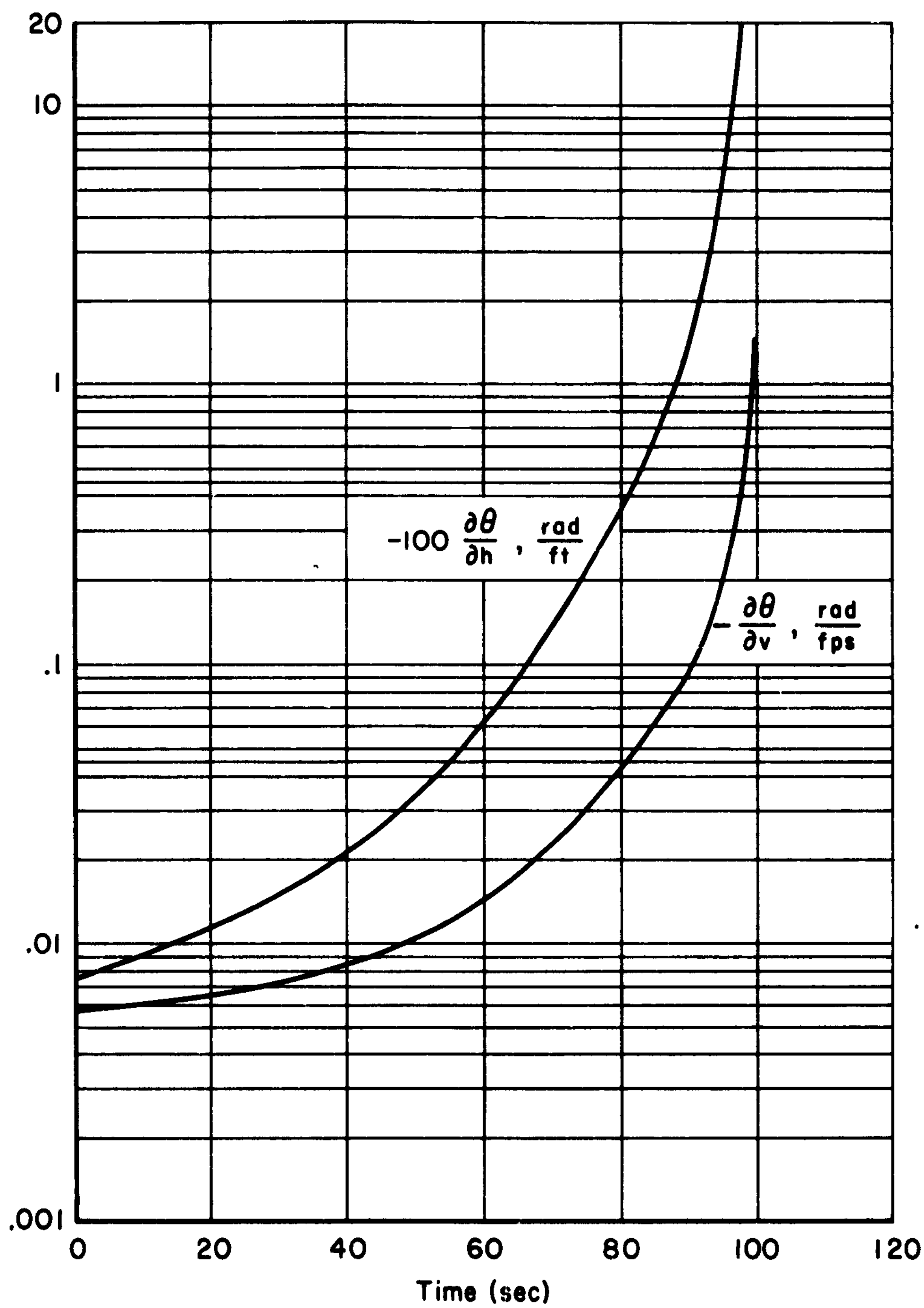

Fig. 1--Time History of Feedback Gains for Idealized Rocket Trajectory Problem

This is analogous to the case of $\delta h = -138$ ft and $\delta v = -92$ ft/sec at $t = 3$ sec, since the nominal trajectory had an altitude of 138 ft and a velocity of 92 ft/sec at time 3. Figure 2 presents a plot of $\delta\theta$ against time for this case. This was the case of most extreme control correction.

It is impossible to precisely evaluate the performance of the guidance scheme. However, the following observations may be made:

1) The terminal altitude in all cases was highly satisfactory.
2) The terminal vertical velocity was such that it would require no more than one quarter of one second additional flight time to drive the vertical velocity to zero. (This is under the assumption that a perturbation of $\delta h = -5000$ would result in a negative v_f of ≈ 6.70 and that the available acceleration was applied in the vertical direction.)
3) The final horizontal velocity was in all cases within 0.1 of one per cent of the optimal horizontal velocity. Note that in some cases where terminal conditions are not quite met "better than optimal" horizontal velocities result.

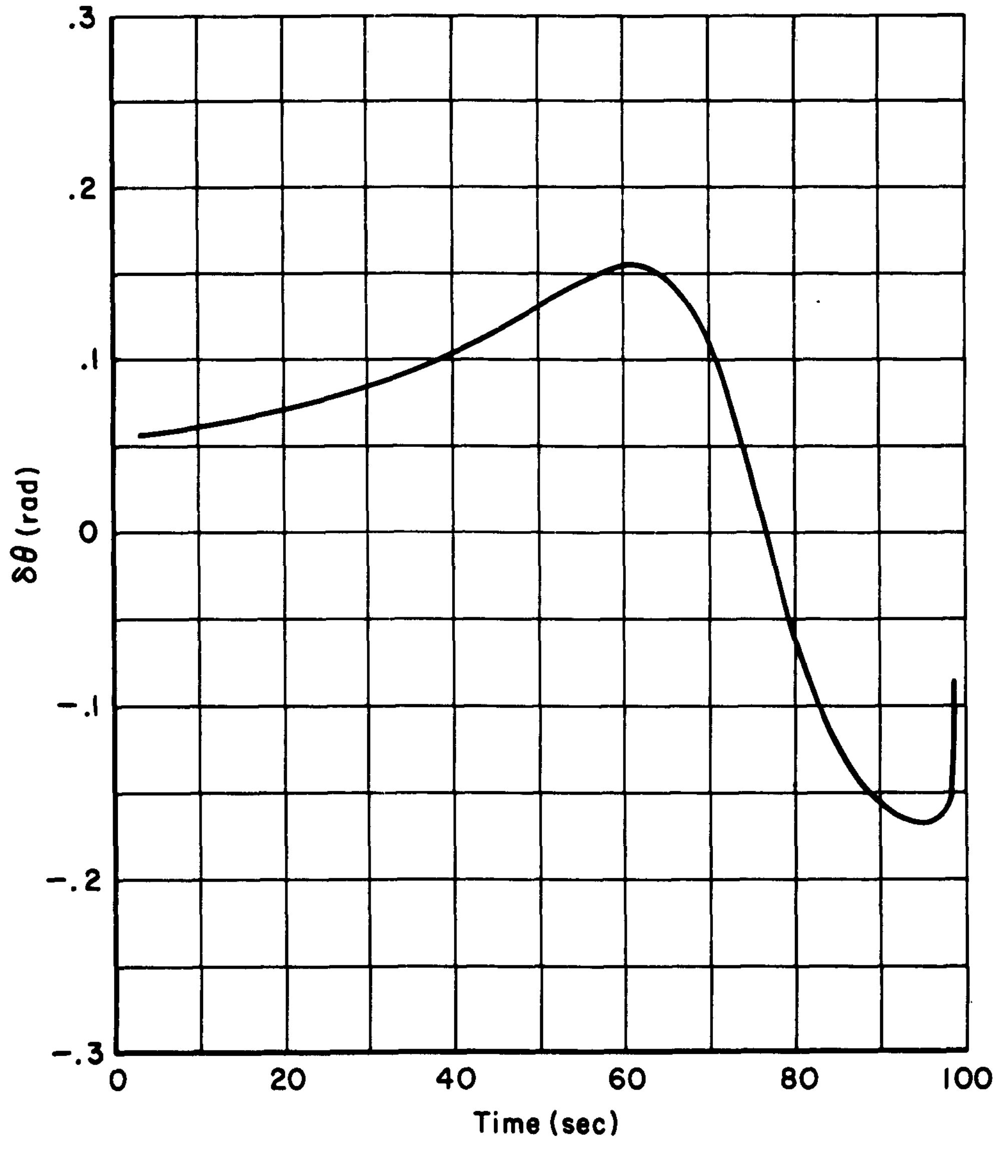

Fig. 2--Time History of Control Correction
for δt = 3 sec Case

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